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Analytic Supernova Models and Black Holes

RICHARD C. ADAMS

Physics Department, University of Pennsylvania
Philadelphia, Pennsylvania 19174
and
Applied Physics Laboratory, Johns Hopkins University
Laurel, Maryland 20810

and

JEFFREY M. COHEN

Physics Department, University of Pennsylvania
Philadelphia, Pennsylvania 19174
and
Mathematical Sciences, Naval Research Laboratory,
Washington, D.C.

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Abstract

Presented are new analytic solutions to Einstein's field equations with properties normally associated with supernovas. These are the first analytic supernova models with pressure, temperature, and luminosity. These solutions are used to compare a radiative nonzero model (for which the pressure is continuous accross the outer boundary of the star) with a radiative zero model [(standard model) for which the pressure within the star is zero at the outer boundary].

1. *Introduction*

The study of the collapse and explosion of a star (an event characterizing a supernova) often requires large complicated computer programs to do the numerical hydrodynamics. There are still many questions that remain to be answered, such as whether neutrinos can eject a significant portion of a star (Colgate and White, 1966; Arnett, 1966; 1967; Wilson, 1971; 1974; Schwartz, 1967) or whether there remains a stable star if the carbon core detonates (Arnett, 1969). Because there are even questions concerning the input data and method of solution (Colgate and Chen, 1972) it would be useful to have an analytic model of a collapsing star even if the characteristics of the model fluid such as equation of state, rate of energy generation, and opacity were not those derived from a theory.

An analytic model could give insights into what phenomena dominate the structure of the star and what is the relative importance of the various characteristics such as temperature, energy generation, and opacity in producing a given luminosity. Such a model could give a zero-order solution with which to investigate the effects of weak shocks in general relativity. At the very least an anlytic model could provide a rigorous check on a complicated computer code.

2. *Equations to be Solved*

In a frame of reference at rest with respect to the matter of the star (called the comoving frame) the metric can be written (Landau and Lifshitz, 1971)

$$dS^2 = -A^2 dt^2 + B^2 dr^2 + R^2 d\theta^2 + R^2 \sin^2\theta d\Phi^2 \tag{2.1}$$

where $G = c = 1$ and the metric parameters A, B, and R are functions of both r and t. This metric differs from those used in other relativistic formulations of hydrodynamics (Wilson, 1971; 1974; Schwartz, 1967) in the choice of the Lagrangian coordinate r instead of the mass. Such a choice allows analytic solutions to be derived. Of course, all formulations must produce equivalent results. The metric in equation (2.1) leads to the Einstein field equations in the comoving frame (Landau and Lifshitz, 1971)

$$8\pi T^{00} = R^{-2} - R_r^2(RB)^{-2} + R_t^2(AR)^{-2} - 2(RB)^{-1}(R_r/B)_r + 2R_tB_t(A^2RB)^{-1} \tag{2.2}$$

$$8\pi T^{11} = 2A_rR_r(AB^2R)^{-1} - 2(AR)^{-1}(R_t/A)_t - R^{-2} + R_r^2(RB)^{-2} - R_t^2(AR)^{-2} \tag{2.3}$$

$$\begin{aligned} 8\pi T^{22} = 8\pi T^{33} &= A_rR_r(AB^2R)^{-1} - R_tB_t(A^2RB)^{-1} - (AR)^{-1}(R_t/A)_t \\ &+ (AB)^{-1}(A_r/B)_r(RB)^{-1} - (AB)^{-1}(B_t/A)_t \end{aligned} \tag{2.4}$$

$$8\pi T^{01} = 2(R_r/B)_t(AR)^{-1} - 2A_rR_t(A^2RB)^{-1} \tag{2.5}$$

where a subscript denotes differentiation with respect to the variable.

To be acceptable the model must have reasonable characteristics such as mass, m, radius $\bar{r}$, and luminosity L_∞, as seen by an outside observer. As shown in Appendix A, the mass, radius, and luminosity in the exterior frame of reference (in terms of variables in the comoving frame evaluated at the outer boundary r_0) is given by (Misner, 1965; Lindquist, 1966)

$$2m = R[1 - R_r^2B^{-2} + R_t^2A^{-2}] \tag{2.6}$$

$$\bar{r}(t) = R(r_0, t) \tag{2.7}$$

$$L_\infty = -R_rM_t(AB)^{-1}[1 + BR_t(AR_r)^{-1}] \tag{2.8}$$

while the time interval, du, of such an external observer is related to dt by (Lindquist *et al.*, 1965; Vadiya, 1951; 1953) [See equation (A3)]

$$du = ABdtR_r^{-1}[1 + BR_t(AR_r)^{-1}]^{-1} \tag{2.9}$$

All the relevant interior quantities, A, B, R and the components of the stress energy tensor can be functions of both r and t. It is a highly nonlinear system of equations, the numerical solution of which taxes the capacity of the biggest and fastest computers.

We will solve these equations with a method used successfully to solve the rotation equation (Adams et al., 1974) and the static field equations (Adams and Cohen, 1975). The method is to invert the usual problem solved in astrophysics. Instead of assuming some form of the equation of state (a definite relationship between the pressure, density, and temperature, usually derived from equilibrium considerations that may not be applicable in a supernova), we find a solution and determine what equation of state induced such a result. The details of the solution are given in Appendix B.

For every general solution to the field equations there are at least two distinct boundary conditions that can be imposed. The first is that the radial components of the stress energy tensor join continuously at the outer stellar boundary. This implies that there is a finite pressure at the outer boundary since there is a finite energy flux carried by neutrinos or photons in the exterior regions. This will be called the radiative nonzero solution. This will be discussed in Appendix B. Another often used condition is the radiative zero (characterized by vanishing pressure at the outer boundary) familiar to astrophysicists because of its use in the standard model (Chandrasekhar, 1939). Having analytic formulas will allow a detailed comparison of the two. Undoubtedly, there are more realistic boundary conditions, but in the absence of a detailed theory of the supernova boundary layers, these two will be used here. Even for nearly adiabatic Newtonian stars, the theory of boundary layers is complicated, and an ambiguous element of the theory of stellar evolution (Schwarzschild, 1958). Presented in the next two sections are models that are essentially identical except for the boundary conditions imposed upon the stress energy tensor at the outer boundary of the star.

3. *A Solution-Radiative Nonzero*

A solution for a star collapsing to a black hole (for which the surface pressure is continuous) is given by (see Appendix B for the derivation)

$$R(r, t) = re^{-\alpha t} \tag{3.1}$$

$$B(r, t) = e^{-\alpha t}[1 - (a_0 + 3a_1 r_0^2)^{2/3} 4a_1 r^2 (a_0 + 3a_1 r^2)^{-2/3} \times (a_0 + 5a_1 r_0^2)^{-1}]^{1/2} \tag{3.2}$$

$$\begin{aligned} A(r, t) = \alpha(a_0 + a_1 r^2) e^{-3\alpha t/2} [\{\alpha + 4a_1 r_0 (a_0 + a_1 r_0^2)^{1/2} \\ \times (a_0 + 5a_1 r_0^2)^{-1/2}\} - e^{-\alpha t/2} 4a_1 r_0 (a_0 + a_1 r_0^2)^{1/2} \\ \times (a_0 + 5a_1 r_0^2)^{-1/2}]^{-1} \end{aligned} \tag{3.3}$$

$$8\pi T^{00} = 4a_1(3a_0 + 5a_1r^2)(a_0 + 3a_1r_0^2)^{2/3} e^{2\alpha t}(a_0 + 3a_1r^2)^{-5/3} \times (a_0 + 5a_1r_0^2)^{-1} + 3[\eta - e^{\alpha t/2}(\eta + q_0)]^2(a_0 + a_1r_0^2)^3 \times e^{2\alpha t}(a_0 + a_1r^2)^{-2}(a_0 + 5a_1r_0^2)^{-1} \tag{3.4}$$

$$8\pi T^{11} = 4a_1e^{2\alpha t}(a_0 + a_1r^2)^{-1}\{1 - (a_0 + 3a_1r_0^2)^{2/3}(a_0 + 5a_1r^2) \times (a_0 + 3a_1r^2)^{-2/3}(a_0 + 5a_1r_0^2)^{-1} + [e^{\alpha t/2}(\eta + q_0) - \eta] \times (a_0 + a_1r_0^2)^2(a_0 + 5a_1r_0^2)^{-1}(a_0 + a_1r^2)^{-1}\} \tag{3.5}$$

$$8\pi T^{01} = 4a_1r(a_0 + a_1r^2)^{-2}[1 - 4a_1r^2(a_0 + 3a_1r_0^2)^{2/3}(a_0 + 5a_1r_0^2)^{-1} \times (a_0 + 3a_1r^2)^{-2/3}]^{1/2}[(q_0 + \eta)e^{\alpha t/2} - \eta]\ (a_0 + a_1r_0^2)^{3/2} \times e^{2\alpha t}r_0^{-1}(a_0 + 5a_1r_0^2)^{-1/2} \tag{3.6}$$

where a_0, a_1, and r_0 are integration constants related to the dimensionless parameters q_0 and η by

$$\eta = 4a_1r_0^2(a_0 + a_1r_0^2)^{-1} \tag{3.7}$$

$$q_0 = \alpha r_0(a_0 + 5a_1r_0^2)^{1/2}(a_0 + a_1r_0^2)^{-3/2} \tag{3.8}$$

The mass and luminosity can be written in terms of the parameter $q(t)$ defined as

$$q = e^{\alpha t/2}(q_0 + \eta)(1 + \eta)^{-1} \tag{3.9}$$

so that

$$2m = r_0e^{-\alpha t}\{4a_1r_0^2 + (a_0 + a_1r_0^2)[\eta - (1 + \eta)q]^2\}(a_0 + 5a_1r_0^2)^{-1} \tag{3.10}$$

$$L_\infty = 2a_1r_0^2(1 + \eta)^2(a_0 + a_1r_0^2)(a_0 + 5a_1r_0^2)^{-2}[q(1 + \eta) - \eta] \times (1 - q)^2 \tag{3.11}$$

The time, u, of a distant external observer also can be expressed in terms of q [equation (2.9)]:

$$u = 2r_0(q_0 + \eta)^2(1 + \eta)^{-3}(a_0 + 5a_1r_0^2)(a_0 + a_1r_0^2)^{-1}F(q, \eta) \tag{3.12}$$

where

$$F(q, \eta) = -\ln(q^{-1} - 1) - q^{-1} - q^{-2}/2 + q^{-2}(\eta^{-1} + 1)/2 + q^{-1}(\eta^{-1} + 1)^2 + (\eta^{-1} + 1)^3 \ln[1 - q^{-1}(\eta^{-1} + 1)^{-1}] \tag{3.13}$$

The effective surface temperature seen by an external observer is related to the total luminosity by (Schwarzschild, 1958)

$$T_e^4 = e^{2\alpha t}L_\infty(\pi a r_0^2)^{-1} = L_\infty(\pi a R^2)^{-1} \tag{3.14}$$

where for photons the constant a is given by

$$a = \pi^2 k^4 \hbar^{-3}/15 \tag{3.15}$$

For neutrinos (fermions), a is 7/8 of the above value.

From equations (3.12) and (3.13) it can be seen that as q approaches 1, u becomes infinite. This occurs at a finite time in the comoving frame as can be seen from equation (3.9). As the time grows large, the luminosity goes to zero exponentially [as can be shown by setting $q = 1 - \epsilon$ in equations (3.11)–(3.13)].

$$L_\infty \to 2a_1 r_0^2 (1+\eta)^2 (a_0 + a_1 r_0^2)(a_0 + 5a_1 r_0^2)^{-2} \exp(-2\beta u) \tag{3.16}$$

where β is

$$\beta = (1+\eta)^3 (a_0 + a_1 r_0^2)(a_0 + 5a_1 r_0^2)^{-1} (q_0 + \eta)^{-2} (2r_0)^{-1} \tag{3.17}$$

The exponential dependence of the luminosity with time in the final stages of collapse has been noted by Thorne and Ames (1968). The mass of the resultant black hole is given by equations (3.9) and (3.10)

$$2m_{BH} = r_0 (q_0 + \eta)^2 (1+\eta)^{-2} \tag{3.18}$$

It is interesting to note that the mass of the black hole depends on the initial conditions. Part of the decrease in the luminosity is due to the doppler shift associated with the velocity of the outer boundary (A35). This is described by the factor $1 + BR_t(AR_r)^{-1}$ that goes like $e^{-\beta u}$ as u becomes large.

4. *A Solution-Radiative Zero*

A solution for a collapsing star characterized by the condition $T^{11} = 0$ at $r = r_0$ with $T^{01} \neq 0$ is given by

$$R(r,t) = re^{-\alpha t} \tag{4.1}$$

$$B(r,t) = e^{-\alpha t}[1 - (a_0 + 3a_1 r_0^2)^{2/3} 4a_1 r^2 (a_0 + a_1 r^2)^{-2/3} (a_0 + 5a_1 r_0^2)^{-1}] \tag{4.2}$$

$$A(r,t) = (a_0 + a_1 r^2) e^{-3\alpha t/2} \tag{4.3}$$

$$8\pi T^{00} = 4a_1 (3a_0 + 5a_1 r^2)(a_0 + 3a_1 r_0^2)^{2/3} e^{2\alpha t} (a_0 + 3a_1 r^2)^{-5/3} \times (a_0 + 5a_1 r_0^2)^{-1} + 3\alpha^2 e^{3\alpha t} (a_0 + a_1 r^2)^{-2} \tag{4.4}$$

$$8\pi T^{11} = 4a_1 e^{2\alpha t}[1 - (a_0 + 3a_1 r_0^2)^{2/3} (a_0 + 5a_1 r^2)(a_0 + 3a_1 r^2)^{-2/3} \times (a_0 + 5a_1 r_0^2)^{-1}] \tag{4.5}$$

The parameter q is related to q_0 defined in equation (3.8) by

$$q = q_0 e^{\alpha t/2} \tag{4.6}$$

The time u as a function of q is given by

$$u = 2\alpha^2 r_0^3 (a_0 + 5a_1 r_0^2)^2 (a_0 + a_1 r_0^2)^{-4} F(q, \eta = 0) \tag{4.7}$$

where $F(q, \eta)$ is defined in equation (3.13). The luminosity and mass are give by

$$L_\infty = 2a_1 r_0^2 q(1-q)(a_0 + 5a_1 r_0^2)^{-1} \tag{4.8}$$

$$2m = 4a_1 r_0^3 e^{-\alpha t}(a_0 + 5a_1 r_0^2)^{-1} + \alpha^2 r_0^3 (a_0 + a_1 r_0^2)^{-2} \tag{4.9}$$

while the mass of the resulting black hole is

$$2m_{BH} = \alpha^2 r_0^3 (a_0 + 5a_1 r_0^2)(a_0 + a_1 r_0^2)^{-3} \tag{4.10}$$

Normally the formulas developed for the radiative zero solutions are simpler than those for the radiative nonzero.

The most striking difference between the two types of solutions is in the asymptotic dependence of the luminosity as a function of time. The luminosity for the radiative zero is a factor of $e^{\beta u}$ larger than the other. In general the radiative zero solution loses more mass, has a greater luminosity, and lasts longer for a given initial mass and radius. The radiative zero requires something like a shell at the outer boundary in order to join smoothly to the exterior Vadiya metric, an implicit assumption of the derivation of the luminosity formula given in equation (2.8). The lack of continuity apparently has a profound effect upon the luminosity observed at infinity.

5. *Definite Models*

Presented is the evolution of two models of supernovas, each of which had an initial mass of 5 $M_\odot$. One of the models, denoted by subscript N is the radiative nonzero solution given in Section 3, while the other, denoted by subscript Z, is the radiative zero solution given in Section 4.

Each of the models had the same initial velocity (1.596×10^9 cm/sec), the same initial radius (2.159×10^8 cm) and the same central density (2.363×10^8 g/cm^3). The central pressure and the total luminosity were different, as were the masses of the resulting black holes. The values for the various constants used were $a_1 = 2.136 \times 10^{-20}$ cm^{-2}, and $\alpha = 7.394$ sec^{-1}, which meant that the initial star was almost Newtonian and of uniform density. The value of the logarithmic decrement (β) of the luminosity was 4.229×10^4 sec^{-1} for the radiative nonzero and 2.413×10^4 sec^{-1} for radiative zero solution.

Figure 1 gives the luminosity versus time of a distant observer of an initial 5 $M_\odot$ model. Note that below the peak of the luminosity, the radiative and nonzero solutions give similar results. The radiative zero model, however, has a peak luminosity 80% larger than the radiative nonzero solution. Past the peak the luminosities drop precipitously because of gravitational red shift and Doppler shift. Because the luminosity is greater for the radiative zero solution, the model loses mass more rapidly than the radiative nonzero solution, as can be seen from Figure 2. From Figure 3 it can be seen that the star contracts rapidly until its radius approaches its gravitational radius. Note that the final radii are different because of the difference in the final masses. Below the

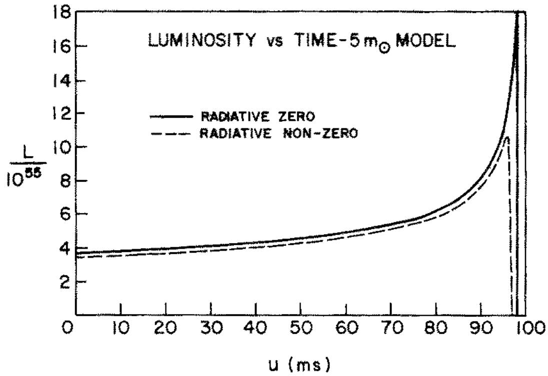

Figure 1. Luminosity observed by distant observer vs. time of a distant observer.

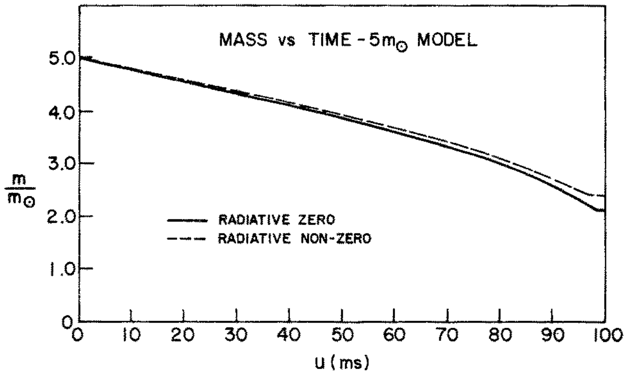

Figure 2. Stellar mass vs. time of distant observer.

peak the radiative zero and nonzero models have very similar effective temperatures, as can be seen from Figure 4. As might be expected, however, from Figure 1, the radiative zero solution has a higher temperature. The equations of state in each case are very nearly polytropic. The adiabatic index is near 2/3 and 5/6 for the radiative zero and nonzero solutions, respectively. It is interesting to note from Figure 5 that the CCLR is between the two other equations of state below 10^{14} g/cm^3.

Table I gives the mass, radius, and luminosity as a function of time to a distant observer. From the table it can be seen that the full width at half-maximum of the luminosity is a factor of 4 larger for the radiative nonzero (23 msec) than the radiative zero model (5.64 msec). After 98.5 msec the radius is close to the gravitational radius (2 m) for each of the models. Table II gives the velocity and comoving time coordinates as a function of distant observer time for the radiative zero and nonzero solutions. Note that

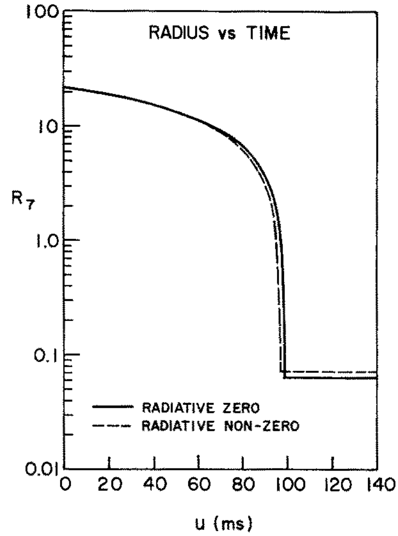

Figure 3. Stellar radius vs. time of distant observer.

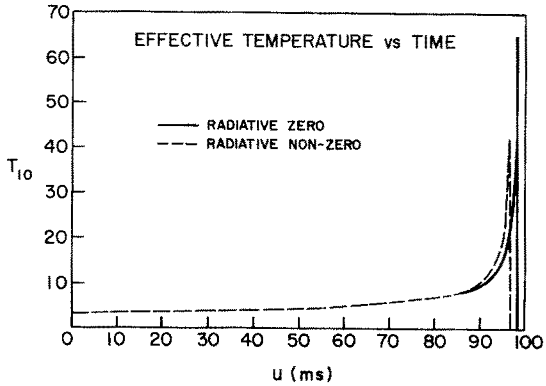

Figure 4. Effective temperature vs. time of distant observer.

for each of the models the velocity of the outer boundary approaches that of light as the gravitational radius is approached.

Also, a number of 100 $M_\odot$ models were treated. It was found that the mass lost due to radiation increased as the equation of state became stiffer.

6. *Discussion*

A striking aspect of the two models is the dramatic drop in the luminosity past 98 msec. This is due to the large values of the logarithmic decrements

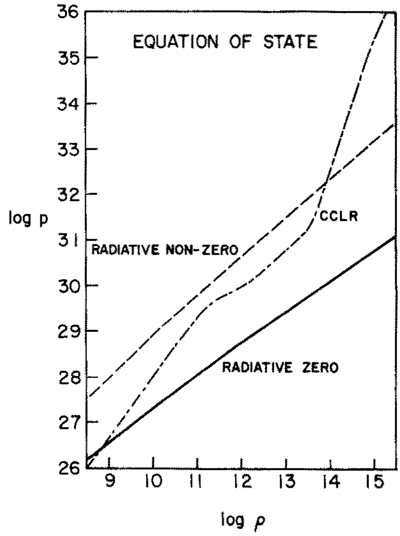

Figure 5. Pressure vs. density–Comparison of equation of state.

given in the previous section. The exponents have time constants of the order of 40 μsec. Thus, an external observer would see one huge flash of light and then darkness from such a star. This is one of the most striking differences between the Newtonian and Einsteinian formulations of hydrodynamics. In the comoving frame (in which Newtonian hydrodynamics is valid to at least 1%) the luminosity continues to increase as it collapses. The shutting off of the luminosity is a distinctively relativistic effect discussed beautifully by Thorne (1969). Owing to radiation, each of these stars loses a significant portion of its initial mass (52% and 58%, respectively) within this 98 msec. After this the luminosity drops to small values. As the star shrinks toward the gravitational radius, the pressure, density, and temperature increase. To a distant observer clocks slow down as the star collapses to a black hole. It may be of interest to note that in the comoving frame 1.8 sec of proper time $[d\tau = A(r = 0, t) \times dt = a_0 S(t) dt]$ elapse from the start of collapse (at $r = 2.159 \times 10^8$ cm) until it reaches its gravitational radius. This is a factor of 21 larger than the free fall time for a collapsing dustball, an effect due to the action of pressure forces as a brake.

7. *Conclusion*

Presented were two distinct stellar models that collapse to black holes. To an outside observer there can be profound differences between the radiative zero and nonzero solutions even for the same initial structure. Such analytic models allow for much greater flexibility in the discussion of final stages of

TABLE I. Mass, radius, and luminosity versus time for two $5M_\odot$ models–Radiative zero and nonzero

u (msec)	$M_N/M_\odot$	$R_N/10^7$	$L_N/10^{55}$	$M_Z/M_\odot$	$R_Z/10^7$	$L_Z/10^{55}$
0	5.00	21.6	3.45	5.00	21.6	3.64
10.0	4.80	20.0	3.57	4.79	20.0	3.77
20.0	4.60	18.5	3.72	4.57	18.4	3.92
29.9	4.39	16.8	3.88	4.35	16.8	4.10
39.9	4.17	15.0	4.09	4.12	15.1	4.31
50.0	3.93	13.1	4.34	3.87	13.2	4.58
55.1	3.80	12.1	4.50	3.73	12.2	4.75
60.0	3.68	11.1	4.68	3.60	11.2	4.94
65.0	3.54	10.0	4.89	3.46	10.2	5.17
70.0	3.40	8.90	5.15	3.31	9.08	5.44
75.0	3.25	7.69	5.47	3.16	7.91	5.79
80.0	3.09	6.39	5.91	2.99	6.65	6.26
85.0	2.92	4.94	6.53	2.80	5.30	6.93
90.0	2.73	3.28	7.58	2.59	3.75	8.05
91.0	2.68	2.89	7.90	2.55	3.40	8.38
92.0	2.64	2.50	8.28	2.50	3.05	8.78
93.0	2.59	2.08	8.75	2.45	2.67	9.27
94.0	2.54	1.61	9.36	2.40	2.27	9.90
94.5	2.51	1.36	9.73	2.37	2.06	10.3
95.0	2.48	1.09	10.1	2.34	1.85	10.8
95.5	2.46	0.792	10.6	2.31	1.61	11.3
95.76	2.44	0.628	10.7	2.29	1.49	11.7
96.0	2.43	0.460	10.4	2.27	1.37	12.0
96.3	2.41	0.240	8.19	2.25	1.22	12.5
96.4	2.41	0.168	5.78	2.25	1.17	12.7
96.5	2.40	0.109	2.20	2.24	1.12	12.9
96.6	2.40	0.0792	0.202	2.23	1.06	13.2
96.7	2.40	0.0721	5.24×10^{-3}	2.22	1.01	13.4
96.8	2.40	0.0711	1.19×10^{-4}	2.22	0.950	13.7
97.0	2.40	0.0709	1.66×10^{-8}	2.20	0.833	14.3
97.2	2.40	0.0709	1.85×10^{-11}	2.18	0.713	14.9
97.4	2.40	0.0709	3.26×10^{-16}	2.17	0.586	15.8
97.5	2.40	0.0709	2.14×10^{-18}	2.16	0.518	16.3
97.7	2.40	0.0709	5.15×10^{-22}	2.14	0.378	17.3
97.88	2.40	0.0709		2.21	0.249	18.0
98.0	2.40	0.0709		2.11	0.157	16.8
98.1	2.40	0.0709		2.10	0.0965	11.5
98.2	2.40	0.0709		2.10	0.0677	3.05
98.3	2.40	0.0709		2.10	0.0624	0.314
98.4	2.40	0.0709		2.10	0.0619	2.93×10^{-2}
98.5	2.40	0.0709		2.10	0.0619	2.47×10^{-3}

TABLE II. Comoving time t, and velocity V versus time u for two $5M_{\odot}$ models–radiative zero and nonzero

u (msec)	t_n (msec)	$\lvert v/c \rvert_N$	t_z (msec)	$\lvert v/c \rvert_z$
0	0	0.0535	0	0.0535
10.0	10.1	0.0557	10.3	0.0557
20.0	21.3	0.0582	21.4	0.0579
29.9	34.0	0.0612	34.1	0.0607
39.9	49.1	0.0651	48.7	0.0641
50.0	67.3	0.0698	66.5	0.0684
55.1	78.1	0.0728	77.2	0.0712
60.0	89.7	0.0763	88.7	0.0743
65.0	104	0.0804	102	0.0780
70.0	120	0.0865	117	0.0826
75.0	140	0.0924	136	0.0845
80.0	165	0.102	159	0.0964
85.0	199	0.116	190	0.108
90.0	255	0.144	237	0.129
91.0	272	0.153	250	0.135
92.0	291	0.165	265	0.143
93.0	317	0.182	283	0.152
94.0	251	0.207	305	0.165
94.5	374	0.225	318	0.173
95.0	404	0.252	333	0.183
95.5	447	0.296	351	0.196
95.76	479	0.334	362	0.204
96.0	521	0.390	373	0.212
96.3	608	0.541	389	0.225
96.4	657	0.648	395	0.230
96.5	715	0.805	401	0.235
96.6	759	0.947	407	0.241
96.7	772	0.993	415	0.248
96.8	773	0.999	422	0.255
97.0	773	1.000	440	0.273
97.2	773	1.000	462	0.295
97.4	773	1.000	488	0.325
97.5	773	1.000	504	0.346
97.7	773	1.000	547	0.404
97.88	773	1.000	604	0.499
98.0	773	1.000	666	0.627
98.1	773	1.000	732	0.800
98.2	773	1.000	780	0.956
98.3	773	1.000	791	0.996
98.4	773	1.000	792	1.000
98.5	773	1.000	792	1.000

the stellar evolution as a function of the initial conditions and the properties of the model fluid than do numerical models.

There are a number of other questions of interest. Are there any more general solutions that are as simple? By the adjustment of some of the initial conditions can such a model become a stable neutron star rather than a black hole? Is there any new physics to be found, i.e., distinctly different physical solutions, within these equations? These questions will be addressed in subsequent papers.

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Appendix A: Matching Metrics across Moving Boundaries

In order to relate physical quantities in one region to those determined in another, the continuity of the first and second fundamental forms can be used. The continuity of these quantities is equivalent to the continuity of the metric coefficients and their normal derivatives at the boundary of the regions (in same reference frame). In an orthonormal Cartan frame, the first and second fundamental forms are the metric and the Ricci rotation coefficients, respectively (Cohen, 1970).

Within the star the metric can be written (Landau and Lifshitz, 1971)

$$ds^2 = -A^2dt^2 + B^2dr^2 + R^2d\Omega^2 \tag{A1}$$

where the angular part is

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\Phi^2 \tag{A2}$$

The metric exterior to a radiating star is (Vadiya, 1951; Misner, 1965; Lindquist *et al.*, 1965)

$$dS^2 = -(1 - 2m\bar{r}^{-1})\,du^2 - 2du d\bar{r} + \bar{r}^2 d\Omega^2 \tag{A3}$$

or

$$ds^2 = \bar{A}^2(du - \bar{A}^{-2}dr)^2 + \bar{A}^{-2}dr^2 + \bar{r}^2 d\Omega^2 \tag{A3a}$$

where

$$\bar{A}^2 = 1 - 2m(u)\bar{r}^{-1} \tag{A4}$$

The parameter $m(u)$ is the total mass as seen by a distant observer while u is the time coordinate of a distant inertial observer at rest with respect to the center of the star. The luminosity as seen by a distant observer (Lindquist et al., 1965; Lindquist, 1966) is given by

$$L_\infty = -dm/du \tag{A5}$$

The metric is continuous at the outer boundary for all θ and Φ if (for $dr = 0$)

$$A^2 dt^2 = \bar{A}^2 du^2 + 2du d\bar{r} \tag{A6}$$

and

$$\bar{r}(t) = R(r_0, t) \tag{A7}$$

Thus, the outer boundary of the star is related in a simple way to the metric coefficient in the comoving frame.

From equation (A3a) a convenient orthonormal frame exterior to the star is

$$\begin{aligned} \bar{\omega}^0 &= \bar{A}du + \bar{A}^{-1}d\bar{r} \\ \bar{\omega}^1 &= d\bar{r}\bar{A}^{-1} \\ \bar{\omega}^2 &= \bar{r}d\theta \\ \bar{\omega}^3 &= \bar{r}\sin\theta d\Phi \end{aligned} \tag{A8}$$

In terms of this frame, the four-velocity and a vector normal to this are given by

$$\bar{u}^\alpha = \gamma(1, v, 0, 0) \tag{A9}$$

$$\bar{\lambda}_\alpha = \gamma(-v, 1, 0, 0) \tag{A10}$$

where

$$\gamma^{-2} = 1 - v^2 \tag{A11}$$

The orthonormal basis forms in a frame exterior to but comoving with the outer boundary are

$$\omega^0 = \gamma(\bar{\omega}^0 - v\bar{\omega}^1) = -\bar{u}_\alpha \bar{\omega}^\alpha \tag{A12}$$

$$\omega^1 = \gamma(\bar{\omega}^1 - v\bar{\omega}^0) = \bar{\lambda}_\alpha \bar{\omega}^\alpha \tag{A13}$$

which is obtained from the relation $\bar{u}_\alpha \bar{\omega}^\alpha = u_\beta \omega^\beta$ and $\bar{\lambda}_\alpha \bar{\omega}^\alpha = \lambda_\beta \omega^\beta$. In the comoving frame the four-velocity is $u^\alpha = (1, 0, 0, 0)$ while the normal is given by $\lambda_i = (0, 1, 0, 0)$. Equations (A12) and (A13) can be inverted to yield

$$\bar{\omega}^0 = \gamma(\omega^0 + v\omega^1) \tag{A14}$$

$$\bar{\omega}^1 = \gamma(\omega^1 + v\omega^0) \tag{A15}$$

The basis forms $\bar{\omega}^2$ and $\bar{\omega}^3$ are equal to their comoving frame counterparts.

To find the Ricci rotation coefficients it is convenient to take the exterior derivative of ω^2 to obtain

$$d\omega^2 = d\bar{r} \wedge d\theta = \bar{r}^{-1}\gamma\bar{A}(\omega^1 + v\omega^0) \wedge \omega^2 \tag{A16}$$

where $d\bar{r}$ has been eliminated in favor of ω^1 and ω^0 in the comoving frame [equations (A15) and (A8)]. Thus, the Ricci rotation coefficient γ^1_{22} in the exterior comoving frame (a quantity that is equal to the derivative in the normal direction) is given by

$$\gamma^{1}_{22} = -\gamma \bar{A} \bar{r}^{-1} \tag{A17}$$

In the interior comoving frame this is given by

$$\gamma^{1}_{22} = -R_r (RB)^{-1} \tag{A18}$$

By equations (A17), (A18), and (A7) and the assumption that the Ricci rotation coefficient is continuous at the boundary gives the relation

$$\gamma \bar{A} = R_r B^{-1} \tag{A19}$$

Another relation of interest is the expression for the proper time and velocity. The computations are facilitated by defining

$$r' = \bar{A}^{-2} d\bar{r}/du \tag{A20}$$

so that equation (A6) can be written

$$d\tau^2 = A^2 dt^2 = \bar{A}^2 du^2 (1 + 2r') \tag{A21}$$

In terms of the orthonormal basis forms u^i is given by (Brill and Cohen, 1966)

$$u^i = \omega^i / d\tau \tag{A22}$$

so that

$$\bar{u}^0 = \gamma = \bar{\omega}^0 / d\tau = (1 + r')(1 + 2r')^{-1/2} \tag{A23}$$

$$\bar{u}^1 = \gamma v = \bar{\omega}^1 / d\tau = \gamma r' (1 + r')^{-1} \tag{A24}$$

From equations (A20), (A21), and (A23) we obtain

$$\bar{A}^2 r'^2 (1 + 2r')^{-1} = R_t^2 A^{-2} \tag{A25}$$

The use of equation (A19) brings the expression for r' into the form

$$r' = BR_t (AR_r)^{-1} [1 - BR_t (AR_r)^{-1}]^{-1} \tag{A26}$$

where all quantities on the right-hand side of equation (4.3) are calculable in the comoving frame from interior quantities. Since all of the desired quantities are expressible in terms of r', we obtain from (A19), (A23), and (A26)

$$\bar{A}^2 = R_r^2 B^{-2} - R_t^2 A^{-2} \tag{A27}$$

from (A21), (A26), and (A27)

$$du = ABdt R_r^{-1} [1 + BR_t (AR_r)^{-1}]^{-1} \tag{A28}$$

from (A5) and (A27)

$$2m = \bar{r}[1 - \bar{A}^2] = R[1 - R_r^2 B^{-2} + R_t^2 A^{-2}] \tag{A29}$$

by differentiating (A29) with respect to u and using equation (A28) we have

$$L_\infty = -dm/du = -R_r m_t (AB)^{-1} [1 + BR_t (AR_r)^{-1}] \tag{A30}$$

and from (A24) and (A26) the expression for the velocity v becomes

$$v = BR_t [AR_r]^{-1} \tag{A31}$$

Thus, the expressions given in equations (2.6)-(2.9) of the paper have been derived. The matching of the solution in one region to that in another allows us to determine what an external observer would see from the events that an observer comoving with the matter would see.

Appendix B

Einstein's field equations in the comoving frame are given in equations (2.2) and (2.5). The mass within the sphere of radius r is given in equation (A29):

$$2m = R[1 - R_r{}^2 B^{-2} + R_t{}^2 A^{-2}] \tag{B1}$$

Differentiation of equation (B1) with respect to t and the use of equations (2.3) and (2.5) yield

$$m_t = 4\pi R^2 [T^{11} R_t + T^{01} A R_r B^{-1}] \tag{B2}$$

In the Vadiya metric, the stress energy tensor in a frame of reference at rest with respect to infinity is given by (using Einstein's field equations for the Vadiya metric)

$$\bar{T}^{00} = \bar{T}^{11} = \bar{T}^{01} = -(4\pi r^2 \bar{A}^2)^{-1} dm/du > 0 \tag{B3}$$

The relation between the stress energy tensor in a frame moving with a velocity v with respect to infinity and the one given above can be found by using the relation for an arbitrary second-rank tensor $K_{\mu\alpha}$

$$\bar{K}_{\mu\alpha} \bar{\omega}^\mu \bar{\omega}^\alpha = K_{\mu\alpha} \omega^\mu \omega^\alpha \tag{B4}$$

The relation between the basis forms is given in equations (A12)-(A15). Hence, the stress energy tensor components are related by

$$T_+^{00} = T_+^{11} = T_+^{01} = T^{01}(1 - v)(1 + v)^{-1} \tag{B5}$$

given in equation (A31). The subscript + on $T_{\mu\alpha}$ denotes quantities evaluated exterior to the star in the comoving frame. The above result is not too surprising in view of the character of electromagnetic radiation. As can be seen from equation (B2) above, if in the interior comoving frame $T_-^{11} = T_-^{01} = T_+^{01}$, then one obtains similar expressions for the luminosity in the interior and exterior of the star corrected in a natural manner for the red-shift and length contraction familiar in relativity. Thus, to obtain consistent results, a condition to be imposed is

$$T^{11} = T^{01} \tag{B6}$$

evaluated at $r = r_0$ in the interior of the star. This condition is equivalent to the continuity of the second fundamental forms, γ^1_{22} given in equation (A18) and γ^1_{00} given by $A_r(AB)^{-1}$, as can be seen from the field equations [equations (B3) and (B5)]. Equation (B6) expresses the continuity of $T^{1\mu}$, the radial components of the stress energy tensor.

To solve equations (B1)-(B4) we assume that A, B, and R are given by

$$R(r, t) = re^{-\alpha t} \tag{B7}$$

$$B(r, t) = B(r)e^{-\alpha t} \tag{B8}$$

$$A(r, t) = A(r)S(t) \tag{B9}$$

where S is a function only of t and A and B are functions only of r. With these assumptions the field equations can be written

$$8\pi T^{00} = e^{2\alpha t}[r^{-2} - (rB)^{-2} + 2B_r(rB^3)^{-1}] + 3\alpha^2 A^{-2} S^{-2} \tag{B10}$$

$$8\pi T^{11} = e^{2\alpha t}[2A_r(rAB^2)^{-1} - r^{-2} + (rB)^{-2}] - A^{-2}[3\alpha^2 S^{-2} + 2\alpha S_t S^3] \tag{B11}$$

$$\begin{aligned} 8\pi T^{22} = e^{2\alpha t}[&A_r(rAB^2)^{-1} + A_{rr}(AB^2)^{-1} - A_r B_r (AB^3)^{-1} \\ &- B_r(rB^3)^{-1}] - A^{-2}[3\alpha^2 S^{-2} + 2\alpha S_t S^{-3}] \end{aligned} \tag{B12}$$

$$8\pi T^{01} = 2\alpha A_r (A^2 B)^{-1} e^{\alpha t} S^{-1} \tag{B13}$$

The usual assumption of astrophysics is that the stress energy tensor for a spherically symmetric system is isotropic, i.e.,

$$T^{11} = T^{22} = T^{33} \tag{B14}$$

This assumption must break down near the surface where radially directed radiation dominates. It is a reasonable assumption, however, and leads to tractable models. In lieu of a detailed theory of the surface layers of a supernova we will use this assumption. We explicitly invoke this assumption by subtracting equation (B11) from (B12) and by requiring that the difference vanish. This leads to what we will call the radial equation

$$0 = -rB_r B^{-3}[A + rAr] + B^{-2}[r^2 A_{rr} - rA_r - A] + A \tag{B15}$$

which can be converted into a first-order linear differential equation for B^{-2} in terms of A (Adams and Cohen, 1975). Two new solutions to this equation are (Adams and Cohen, 1975)

$$A = a_0 + a_1 r^2 \tag{B16}$$

$$B^{-2} = 1 - f_0 r^2 (a_0 + 3a_1 r^2)^{-2/3} \tag{B17}$$

and

$$A = a_0 \exp(a_1 r^2) \tag{B18}$$

$$B^{-2} = 1 - r^2 e^{2a_1 r^2}[f_0 + 2a_1 e^{-1} Ei(1 + 2a_1 r^2)] \tag{B19}$$

where Ei(x) is the exponential integral (Abramowitz and Stegun, 1964). The first solution is the one used in the paper. The constant f_0 can be adjusted so that the bracketed quantity multiplying the $e^{2\alpha t}$ in T^{11} is zero at $r = r_0$. The result is $f_0 = 4a_1(a_0 + 3a_1r_0^2)^{2/3}(a_0 + 5a_1r_0^2)^{-1}$ (B20)

It remains now only to impose a boundary condition upon the other term. The radiative zero boundary condition is obtained by setting this term equal to zero, i.e.,

$$2S_t = -3\alpha S \tag{B21}$$

or

$$S(t) = e^{-3\alpha t/2} \tag{B22}$$

These assumptions lead to the radiative zero solution given in equations (4.1)–(4.9) in the paper.

The radiative nonzero solution is obtained by requiring that the extra term in equation (B11) or (B12) be equal to the term in equation (B13) for T^{01} at $r = r_0$. Thus, the boundary condition that $T^{11} = T^{01}$ at $r = r_0$ is satisfied if

$$2A_rB^{-1}e^{\alpha t}S^{-1} = -2S_tS^{-3} - 3\alpha S^{-2} \tag{B23}$$

The results of the solution of the above equation are given in the radiative nonzero solution in equations (3.1)–(3.13) of the paper.

Thus, two different boundary conditions applied to the stress energy tensor can have profoundly different effects as seen by an observer at infinity. It should be noted that the radiative nonzero boundary condition (because of its internal consistency and continuity) is the more natural and realistic. Yet, the radiative zero boundary condition (because of its simplicity) is often used in the study of supernova events.

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